Toppling distributions in one-dimensional Abelian sandpiles

1992 J. Phys. A: Math. Gen. 25 L1257

(http://iopscience.iop.org/0305-4470/25/22/006)

J. Phys. A: Math. Gen. 25 (1992) L1257–L1264. Printed in the UK

LETTER TO THE EDITOR

Toppling distributions in one-dimensional Abelian sandpiles

Philippe Ruelle† and Siddhartha Sen‡

† Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

‡ School of Mathematics, Trinity College Dublin, 18 Westland Row, Dublin 2, Ireland

Received 18 August 1992

Abstract. We consider toppling distributions for the Abelian sandpile model in one dimension. We study the avalanche mass and duration distributions in the thermodynamic limit. We also investigate their dependence on the seeding distribution, which describes how sand is dropped. None of the models shows criticality.

Over the past few years, sandpile models have received much attention as they are among the simplest models showing self-organized criticality (SOC). The concept of SOC was introduced by Bak, Tang and Wiesenfeld as an attempt to explain the occurrence of power laws in various and many natural phenomena [1]. These authors suggested that the dynamics involved may be such that it drives the system to criticality, without the adjustment of any parameter. In some sense, critical points would be attractors for the dynamics.

Sandpile models are discrete models defined on a lattice and possess a cellular automaton type of dynamics. A general analysis of the critical height models was undertaken by Dhar [2]. He showed that the sandpile automaton features an Abelian group (hence the name Abelian sandpiles) and obtained a simple characterization of its critical state. Intensive numerical study, mainly of two-dimensional models, has provided much numerical data, including critical exponents [3–7]. On the theoretical side, much progress has been done and exact (though partial) results are available [2, 8–11]. They show that in dimension $d \geqslant 2$, the sandpile models are critical, in the restricted sense that they exhibit power laws. In one dimension, the SOC state (we keep the name although the model does not show criticality) is simple enough to allow very explicit calculations. While correlation functions are trivial, we show in this letter that the toppling distributions, though not critical, are not trivial.

We consider a linear chain of length L, with sites numbered 1 to L. To each site i, we assign a variable z_i taking the values 0 and 1. (z_i is the 'height' of the sandpile at i.) A stable configuration C is a set of values z_i, $i = 1, \ldots, L$. We denote by S the space of all stable configurations, in number equal to 2^L. The dynamics of the model is defined as follows.

(i) With probability p_j, add a 'grain of sand' at site j, that is $z_j \longrightarrow z_j + 1$ and $z_i \longrightarrow z_i$ for $i \neq j$.

(ii) If some site i has $z_i \geqslant 2$, it is critical and topples. In doing so, its height is reduced by two units of sand and each of its neighbours receives one unit: $z_i \longrightarrow z_i - 2, z_{i\pm 1} \longrightarrow z_{i\pm 1} + 1, z_k \longrightarrow z_k$ if $k \neq i, i \pm 1$. When all the critical sites have toppled, we are left with a new stable configuration.

Note that under toppling, the quantity of sand is conserved, except if one of either end sites topples. In that case a grain of sand falls off the lattice. A toppling matrix Δ is usually introduced which specifies the way sand is redistributed when a toppling occurs: under toppling of site i, $z_k \longrightarrow z_k - \Delta_{ik}$. In our case,

$$\Delta_{ik} = \begin{cases} 2 & \text{if } i = k \\ -1 & \text{if } |i-k| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

which is nothing but the discrete Laplacian in one dimension. It is also convenient to define operators a_j, $j = 1, \ldots, L$, which carry out the two aforementioned steps: a_j adds one grain at site j and let the configuration relax to a new stable configuration. (The a_j generate the Abelian group alluded to earlier.)

When the dynamics is applied to some initial configuration, only the recurrent configurations keep occurring. They are those configurations which are obtained by acting with all the a_j on the configuration $C^* = \{z_i = 1 \text{ for all } i\}$. We denote by $\mathcal{R}$ the space of recurrent configurations. The SOC state of the sandpile is the stationary state: it assigns a zero probability to a non-recurrent configuration, and is a uniform distribution on $\mathcal{R}$ [2].

Each time one of the operators a_j is applied, a certain number of topplings take place before we return to a stable configuration. In addition the avalanche takes some time before it settles down. These two quantities, the mass and the duration of an avalanche, are random numbers. Here we compute their probability distribution in the SOC state.

For the one-dimensional sandpile model, the space $\mathcal{R}$ can be very explicitely given: it has cardinality $L+1$ and contains all the configurations with at most one height equal to 0. They can be all written as $a^{L+1-l}C^* \equiv a_1^{L+1-l}C^*$, with $0 \leqslant l \leqslant L$. The configuration $a^{L+1-l}C^*$ has no 0 if $l = 0$ ($a^{L+1}C^* = C^*$), and has one 0 at position l otherwise:

$$z_i(a^{L+1-l}C^*) = 1 - \delta_{i,l} \qquad 1 \leqslant i \leqslant L. \tag{2}$$

Seeding at site j is implemented by the operator a_j, and satisfies $a_j = a_1^j = a^j$, so that the group generated by the a_j is cyclic of order $L+1$. Because each of the recurrent configurations has probability $1/(L+1)$ in the SOC state, it is a trivial matter to obtain the correlation functions ($i_1 \neq i_2 \neq \cdots \neq i_r$)

$$\langle (z_{i_1} - \langle z \rangle)(z_{i_2} - \langle z \rangle) \ldots (z_{i_r} - \langle z \rangle) \rangle_{\mathcal{R}} = -\frac{r-1}{(L+1)^r} \longrightarrow 0 \qquad L \longrightarrow \infty. \tag{3}$$

Following Creutz [9], we define the quantity $T_{ij}(C)$, which is the number of topplings occurring at site i during the relaxation of $a_j C$. $T_{ij}(C)$ satisfies the following equation, valid for any stable configuration C [9]

$$z_i(a^j C) = z_i(C) + \delta_{i,j} - \Delta_{ik}\, T_{kj}(C). \tag{4}$$

For $C = a^{L+1-l}C^*$ in $\mathcal{R}$, by using (2) and the inverse $(\Delta^{-1})_{ij} = \min(i,j) - ij/(L+1)$, we obtain the following explicit form of $T_{ij}(C)$, symmetric in i and j:

$$T_{ij}(a^{L+1-l}C^*) = \begin{cases} \min(i,j)+\min(i+j,l)-i-j & \text{if } i,j<l \\ \min(i,j)+\min(i+j,L+1-l)-(i+j+l) & \text{if } i,j>l \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Let S be the random variable counting the number of topplings (or mass of the avalanche) per added particle, computed in the space of recurrent configurations. The probability to observe exactly s topplings when one grain of sand is dropped randomly, with a uniform distribution ($p_j = 1/L$) is given by

$$\begin{aligned} \text{Prob}\,[S=s] &= \sum_{j=1}^{L} p_j \,\text{Prob}\left[\sum_i T_{ij} = s\right] \\ &= \frac{1}{L(L+1)} \sum_{j=1}^{L} \sum_{l=0}^{L} \delta\left(\sum_{i=1}^{L} T_{ij}(a^{L+1-l}C^*) = s\right). \end{aligned} \tag{6}$$

From (5), we obtain the total number of topplings when $a^{L+1-l}C^*$ relaxes

$$\sum_{i=1}^{L} T_{ij}(a^{L+1-l}C^*) = \begin{cases} j(l-j) & \text{if } \leqslant l \\ (L+1-j)(j-l) & \text{if } j \geqslant l. \end{cases} \tag{7}$$

It is clear from (7) that $\text{Prob}\,[S=s]$ will strongly depend on the factorization properties of s. We obtain from (6) and (7)

$$\text{Prob}\,[S=s] = \begin{cases} \dfrac{1}{L+1} & \text{for } s=0 \\ \dfrac{1}{L(L+1)}[2\,\text{card}\,\{d : d \mid s \text{ and } d+s/d \leqslant L\} \\ \qquad + \text{card}\,\{d : d \mid s \text{ and } d+s/d = L+1\}]. \end{cases} \tag{8}$$

The value at $s=0$ is clear since there are L configurations which do not produce any toppling at all if their zero-height site is seeded. We note that the value $s=0$ is always more likely than any other value. Let us also note that the maximal value of S is

$$s_{\max} = \begin{cases} \frac{1}{4}(L(L+2)) & \text{if } L \text{ is even} \\ \frac{1}{4}((L+1)^2) & \text{if } L \text{ is odd.} \end{cases} \tag{9}$$

This maximum is reached when the minimally stable configuration C^* is seeded at its centre, or at one of the two most central sites if L is even.

The moments of S are easily computed. The first two read

$$\langle S \rangle = \frac{(L+1)(L+2)}{12} \tag{10a}$$

$$\langle S^2 \rangle = \frac{(L+1)(L+2)(2L^2+4L+9)}{180} \tag{10b}$$

while the others can be computed in terms of the Bernouilli polynomials, with leading term

$$\langle S^m\rangle = \frac{(m!)^2}{(m+1)(2m+1)!}L^{2m} + \mathcal{O}(L^{2m-1}) \qquad m \geqslant 0. \tag{11}$$

Let us now consider the infinite-L limit of the renormalized random variable S/L^α. If $\alpha > 2$ or $\alpha < 2$, the distribution of the renormalized variable is trivial, being a delta function at zero or at infinity respectively. For $\alpha = 2$ however, the limit is non-trivial. Let us define $Y = \lim_{L\to\infty} S/L^2$. From (9) and (11), it is clear that Y is a continuous random variable taking its values between 0 and $\frac{1}{4}$, and that all its moments are finite:

$$\langle Y^m\rangle = \frac{(m!)^2}{(m+1)(2m+1)!}. \tag{12}$$

By Carleman's theorem [12], the moments (12) define a unique probability distribution.

It is straightforward to compute the density function of Y. The characteristic function is

$$\chi_Y(t) = \langle \mathrm{e}^{\mathrm{i}tY}\rangle = \frac{2}{\mathrm{i}t}\int_0^1 \frac{\mathrm{d}x}{x\sqrt{1-x}}(\mathrm{e}^{\mathrm{i}tx/4} - 1) \tag{13}$$

from which the density function $f_u(y)$ (corresponding to uniform seeding) is recovered by inverse Fourier transform

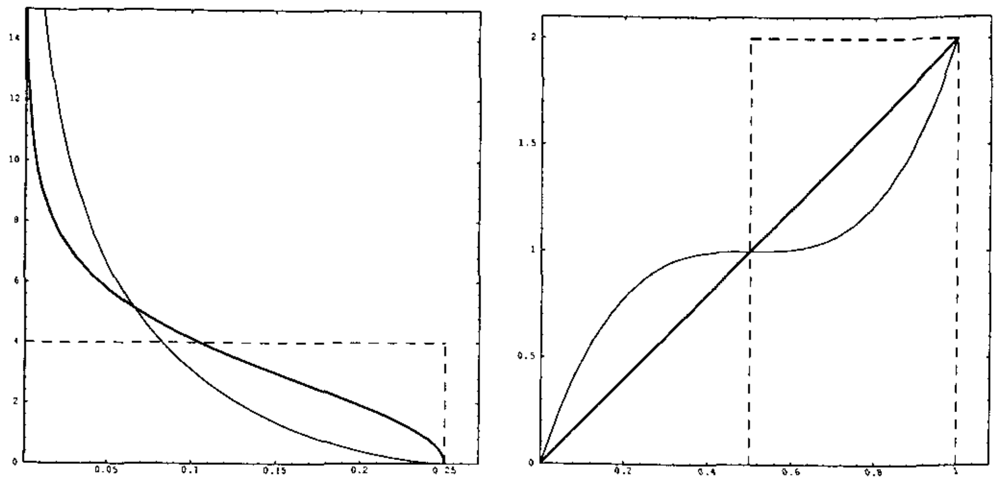
$$f_u(y) = \begin{cases} 2\log\dfrac{1+\sqrt{1-4y}}{1-\sqrt{1-4y}} & \text{if } 0 \leqslant y \leqslant \frac{1}{4} \\ 0 & \text{otherwise.}\end{cases} \tag{14}$$

The function f_u has a logarithmic singularity at $y = 0$, a remnant of the high probability of observing no toppling at all. At $y = \frac{1}{4}$, $f_u(y)$ vanishes like a square root $f_u(y) \sim 4\sqrt{1-4y}$. The shape of f_u is pictured in figure 1.

Table 1. Density functions for the avalanche mass and duration distributions in the thermodynamic limit, depending on the seeding distribution. The functions are zero outside the range showed on the top line.

Seeding dist.	Avalanche mass ($0 \leqslant y \leqslant \frac{1}{4}$)	Duration ($0 \leqslant t \leqslant 1$)
Uniform	$f_u(y) = 2\log\dfrac{1+\sqrt{1-4y}}{1-\sqrt{1-4y}}$	$g_u(t) = 2t$
Binomial	$f_b(y) = 4$	$g_b(t) = 2I_{[\frac{1}{2},1]}(t)$
Quadratic	$f_q(y) = 6\log\dfrac{1+\sqrt{1-4y}}{1-\sqrt{1-4y}} - 12\sqrt{1-4y}$	$g_q(t) = 6t - 12t^2 + 8t^3$

One may wonder whether the asymptotic distribution (14) is robust with respect to the seeding distribution p_j. In order to investigate this question, we repeat these steps for other distributions. The moments of S can be given in all generality. From (7),



Figures 1 and 2. Graphs of the toppling distributions in the SOC state, namely the avalanche mass (left) and the avalanche duration (right), as given in table 1. The thick full curves relate to the uniform seeding distribution, the thin full curves to the quadratic and the broken lines to the binomial.

they can be written as an expectation value with respect to the seeding distribution p_j, provided this one is independent of the configuration being seeded:

$$\langle S^m\rangle = \frac{1}{(L+1)(m+1)}\langle j^m B_{m+1}(L+1-j) + (L+1-j)^m B_{m+1}(j+1) - j^m B_{m+1} - (L+1-j)^m B_{m+1}\rangle_{\text{seeding}} \tag{15}$$

where $B_{m+1}(x) = x^{m+1} + \cdots$ is the $(m+1)$th Bernouilli polynomial and $B_{m+1} = B_{m+1}(0)$ is the $(m+1)$th Bernouilli number [13].

We will use the same method as for the uniform seeding distribution. The expression (15) is usually a polynomial in L (in the limit of large L), of which we pick out the highest power, which is L^{2m} if the kth moment of p_j scales like L^k. We set $\langle j^k\rangle_{\text{seeding}} = a_k L^k + O(L^{k-1})$ and $\langle S^m\rangle = c_m L^{2m} + O(L^{2m-1})$. In addition to the uniform distribution $p_j = 1/L$, we consider the following two: the binomial distribution, which gives more weight to the centre of the lattice, and a centred quadratic distribution, which, on the contrary, privileges the two edges. We include the earlier data about the uniform distribution:

uniform:

$$p_j = \frac{1}{L} \qquad a_k = \frac{1}{k+1} \qquad c_m = \frac{B(m+1,m+1)}{m+1}$$

binomial:

$$p_j = 2^{-(L-1)}\binom{L-1}{j-1} \qquad a_k = 2^{-k} \qquad c_m = \frac{4^{-m}}{m+1} \tag{16}$$

quadratic:

$$p_j = \frac{12((L+1)/2-j)^2}{L(L^2-1)} \qquad a_k = \frac{3(k^2+k+2)}{(k+1)(k+2)(k+3)}$$

$$c_m = \frac{6B(m+2,m+2)}{(m+1)^2}$$

where $B(m+1, n+1) = m!n!/(m+n+1)!$ is the Euler beta function.

In all three cases, we define the renormalized variable $Y = \lim_{L\to\infty} S/L^2$. Clearly, the moments of Y are just the coefficients c_m displayed in (16), $\langle Y^m \rangle = c_m$. In each case, we compute the characteristic function of Y and take its inverse Fourier transform to obtain the density function $f(y)$, noted respectively by f_u, f_b and f_q. The results are given in table 1, and in figure 1.

The various distributions can be well understood from (7). When a configuration is seeded at site j, the total number of topplings s is small if j is close to 1 or to L, while a value of j around $L/2$ gives a maximum number of topplings. The binomial distribution favours large values of s and suppresses small values of s. Therefore in the thermodynamic limit, when compared to the uniform distribution, the binomial smoothes out the logarithmic singularity at $y = 0$ of f_u and takes the curve up at $y = \frac{1}{4}$, the result being just a uniform distribution, $f_b(y) = 4I_{[0,1/4]}(y)$. The quadratic distribution does the converse, and so f_q keeps the singularity at $y = 0$, but at $y = \frac{1}{4}$, it vanishes much faster than f_u, namely $f_q(y) \sim 4(1-4y)^{3/2}$.

We have also considered a deterministic seeding at $j = 1$, that is $p_j = \delta_{j,1}$, which puts an exaggerate weight on the small values of s. In fact, so much that the maximum value $s_{\max}$ is L instead of $L^2/4$. This combined with the value $\langle j^k \rangle_{\text{seeding}} = 1$ (no scaling in L) produces a uniform distribution on $[0, 1]$ for $Y = \lim_{L\to\infty} S/L$.

We now turn to the avalanche duration distributions. The duration will be defined as the number of times the lattice needs to be swept before the sandpile settles in a stable configuration. If the seeding at j produces no toppling at all ($z_j = 0$), the duration is zero. Take for instance $L = 6$ and the configuration (1,1,1,1,0,1) seeded at $j = 2$. The duration is 4: $(1,2,1,1,0,1) \longrightarrow (2,0,2,1,0,1) \longrightarrow (0,2,0,2,0,1) \longrightarrow (1,0,2,0,1,1) \longrightarrow (1,1,0,1,1,1)$.

It is easy to see that if the recurrent configuration $a^{L+1-l}C^*$ is seeded at site j, the duration d_j of the avalanche is $(0 \leqslant l \leqslant L)$

$$d_j(a^{L+1-l}C^*) = \begin{cases} l-1 & \text{if } j < l \\ 0 & \text{if } j = l \\ L-l & \text{if } j > l. \end{cases} \tag{17}$$

Let D be the random variable giving the duration of an avalanche per added particle (in the space of recurrent configurations). We first work out the case of a uniform seeding distribution, $p_j = 1/L$. From (17), D takes its values between 0 and L, with a probability given by

$$\begin{aligned} \text{Prob}\,[D = d] &= \frac{1}{L(L+1)} \sum_{j=1}^{L} \sum_{l=0}^{L} \delta(d = d_j) \\ &= \begin{cases} \dfrac{1}{L+1} & \text{for } d = 0, L \\ \dfrac{2d}{L(L+1)} & \text{for } 1 \leqslant d \leqslant L-1. \end{cases} \end{aligned} \tag{18}$$

It is straightforward to compute the moments of D. For large m, they scale like L^m: $\langle D^m \rangle = 2/(m+2)L^m + \cdots$, which suggests defining the renormalized variable $T = \lim_{L\to\infty} D/L$. The values of T run over $[0, 1]$ and its moments are

$\langle T^m \rangle = 2/(m+2)$. We then find that the density function $g_u(t)$ of T is linear

$$g_u(t) = \begin{cases} 2t & \text{for } 0 \leqslant t \leqslant 1 \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Unlike the distribution for the number of topplings, the duration distribution is not very sensitive to the seeding distribution, as we might have expected. Similarly to (15), the moments of the finite L variable can be computed for a general seeding distribution, with the result

$$\langle D^m \rangle = \frac{1}{(L+1)(m+1)} \langle B_{m+1}(L) + B_{m+1}(L+1) - B_{m+1}(j) - B_{m+1}(L+1-j) \rangle_{\text{seeding}}. \tag{20}$$

In the large L limit, the equation (20) simplifies to $\langle D^m \rangle = 2L^m/(m+1)[1 - \langle j^{m+1} \rangle_{\text{seeding}}]$ in case the seeding distribution is symmetric around the centre of the lattice, $p_j = p_{L+1-j}$. From the moments of p_j displayed in (16), we obtain the moments for the renormalized variable $T = \lim_{L\to\infty} D/L$

$$\begin{aligned} \langle T^m \rangle_{\text{binomial}} &= \frac{2}{m+1}(1 - 2^{-(m+1)}) \\ \langle T^m \rangle_{\text{quadratic}} &= 2\left[\frac{3}{m+2} - \frac{6}{m+3} + \frac{4}{m+4}\right]. \end{aligned} \tag{21}$$

The corresponding density functions $g_b(t)$ and $g_q(t)$ are polynomials in t, constant for g_b and cubic for g_q. They are shown in table 1 and in figure 2.

Finally, it is interesting to compare the distributions computed in the space of recurrent configurations with the analogous quantities, computed in the space of all stable configurations, for which each site assumes the values 0 and 1 with equal probability and independently of the other sites. We restrict ourselves to a uniform seeding distribution.

We omit the details and merely give the results. The variables S and D are defined as earlier. The striking difference is that even when L goes to infinity, the moments of S and D remain finite. For example the first moment of S reads

$$\langle S \rangle = 2 - \frac{4}{L} + \frac{L+4}{L} 2^{-L}. \tag{22}$$

As a consequence, we can define $Y = \lim_{L\to\infty} S$ and $T = \lim_{L\to\infty} D$, without any renormalization, so that in the limit, the random variables Y and T remain discrete and take their values in Z_+. We find that the density function of Y is

$$\text{Prob}\,[Y = y] = \begin{cases} \frac{1}{2} & \text{for } y = 0 \\ \frac{1}{2} \sum_{\substack{k \geqslant 1 \\ k \mid y}} 2^{-(k + y/k)} & \text{for } y \geqslant 1 \end{cases} \tag{23}$$

while that of T is simpler

$$\text{Prob}\,[T = t] = \begin{cases} \frac{1}{2} & \text{for } t = 0 \\ \frac{1}{4} t\, 2^{-t} & \text{for } t \geqslant 1. \end{cases} \tag{24}$$

Due to its arithmetic nature, the function Prob $[Y = y]$ has no asymptotic behaviour. Hence we consider the distribution function Prob $[Y \geqslant n]$, and the same function for T. We find

$$\mathrm{Prob}\,[Y \geqslant n] \sim 2\sqrt{n}K_1(2\sqrt{n}\ln 2) \sim \sqrt{\frac{\pi}{\ln 2}}n^{1/4}\mathrm{e}^{-2\sqrt{n}\ln 2} \qquad n \text{ large} \tag{25}$$

and

$$\mathrm{Prob}\,[T \geqslant n] \sim \tfrac{1}{2}n\mathrm{e}^{-n\ln 2} \qquad n \text{ large}. \tag{26}$$

It is intriguing to note that they are close to satisfy the scaling relation $T = \sqrt{Y}$.

In conclusion, it is somewhat remarkable that the distributions show non-trivial functional form, despite the simplicity of the model. We found that the duration distribution is not very sensitive to the seeding distribution, unlike the mass distribution which is more sensitive. This can be compared with the two-dimensional situation where neither distribution is expected to be very dependent on the seeding [7].

While completing the manuscript, we received a work by F Pythoud [14] which partly overlaps the results presented here. We would like to thank Deepak Dhar for comments on the manuscript and for sending us [14]. PR also wishes to thank Anatoly Patrick for helpful discussions.

References

[1] Bak P, Tang C and Wiesenfeld K 1988 *Phys. Rev. Lett.* **59** 381; 1988 *Phys. Rev.* A **38** 364
[2] Dhar D 1990 *Phys. Rev. Lett.* **64** 1613
[3] Tang C and Bak P 1988 *Phys. Rev. Lett.* **60** 2347
[4] Kadanoff L, Nagel S, Wu L and Zhou S 1989 *Phys. Rev.* A **39** 6524
[5] Manna S 1990 *J. Stat. Phys.* **59** 509
[6] Ohtsuki T and Iwata H 1992 *Phys. Rev.* A **45** 2228
[7] Zhang Y 1989 *Phys. Rev. Lett.* **63** 470
[8] Dhar D and Majumdar S 1991 *J. Phys. A: Math. Gen.* **23** 4333
[9] Creutz M 1991 *Comput. Phys.* **5** 198
[10] Majumdar S and Dhar D 1991 *J. Phys. A: Math. Gen.* **24** L357; 1992 *Physica* **185A** 129
[11] Priezzhev V 1992 *Preprint* Dublin DIAS-STP-92-10
[12] Feller W 1966 *An Introduction to Probability Theory and Its Applications* vol II (New York: Wiley)
[13] Gradshteyn I and Ryzhik I 1965 *Table of Integrals, Series and Products* (New York: Academic)
[14] Pythoud F 1992 *Diploma Thesis* ETH Zürich